The metaphor behind the generalizing BG Entropy

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1. The first one is find a function such that its slope is zero everywhere.

$$\frac{dy}{dx} = 0 \Rightarrow \text{that means } y \text{ and } x \text{ are constants under the constant } (y(0) = 1)$$

2. And a function that, its slope is constant everywhere. Let it be 1, so it is like $y = x + c$

$$\frac{dy}{dx} = 1, \quad (y(0) = 1) \Rightarrow y = 1 + x$$

3. Find a function such that its slope is equal to itself everywhere. Let's solve why it is?

$$\begin{aligned}
\frac{dy}{dx} = e^x \Rightarrow \frac{dy}{dx} &= y \\
&\Rightarrow (1/y)dy = dx \\
&\Rightarrow \int (1/y)dy = \int dx \\
&\Rightarrow \ln(y) = x + c \\
&\Rightarrow e^{\ln(y)} = e^{x+c} \\
y &= e^x \cdot e^c
\end{aligned}$$

Let $e^c = A$, for $y(0) = 1$:

$$\begin{aligned}
1 &= e^0 \cdot A \Rightarrow A = 1 \\
y &= e^x \Rightarrow y = \ln x
\end{aligned}$$

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So let them together:

$$\frac{dy}{dx} = 0$$

Static world, main idea is system is constant.

$$\frac{dy}{dx} = 1$$

$\Rightarrow$ the change rate is constant and it is 1. Main idea is the universe is going to expanding at constant rate from outside factor.

$$\frac{dy}{dx} = y$$

$\Rightarrow$ change rate depends on system's current situation. This means, the system is growing by itself, no addition from outside factor.

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Now the question is, can we unify these equations? Are these equations would be in form that $\frac{dy}{dx} = a + by$? $(y(0) = 1)$

$$\frac{dy}{dx} = 0 \Rightarrow a = 0, b = 0 \Rightarrow dy/dx = 0$$
$$\frac{dy}{dx} = 1 \Rightarrow a = 1, b = 0 \Rightarrow dy/dx = 1$$
$$\frac{dy}{dx} = y \Rightarrow a = 0, b = 1 \Rightarrow dy/dx = y$$

So this unifying depends on two parameters. Now the question is can we write it as a single parameter form?

$$\frac{dy}{dx} = y^q, \quad (y(0) = 1), \quad (q \in \mathbb{R}) \ ?$$

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Let's solve 3

1)
$$dy/dx = y^q$$

2)
$$(1/y^q)dy = dx$$

3)
$$y^{-q}dy = dx$$

Now take the integral;

4)
$$\frac{y^{(-q+1)}}{(-q+1)} = x + c$$

5) Now let's solve c constant;

$$x = 0, y = 1 \Rightarrow \frac{1^{(-q+1)}}{(-q+1)} = 0 + c$$
$$\frac{1}{(1-q)} = c$$

6) Rewrite into equation;

$$\frac{y^{(1-q)}}{(1-q)} = x + \left(\frac{1}{(1-q)}\right)$$

7) Multiply both side by $(1-q)$

$$y^{(1-q)} = (1-q)x + 1$$

8) Take the exponent $(1/(1-q))$ on both side

9)

$$y = [1 + (1-q)x]^{\frac{1}{(1-q)}}$$

Let's evaluate what we found;

$$\frac{dy}{dx} = 1 \Rightarrow \text{Second Equation previously our world scenario formulation.}$$

(Put $x = 0, y = 1$ into equation or solve it for $q = 0$)
For $q = 1$, it is multiplicative world.

$$\frac{dy}{dx} = y \Rightarrow \text{third equation we previously wrote for generalization.}$$

This part a bit problematic because there is a divider "zero" let's solve it $\Rightarrow$

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Let's take the limit $\Rightarrow$

$$\lim_{n\to\infty} (1 + x/n)^n = e^x$$

So $q = 1$ only be for $n \to \infty$ our formula $y = [1 + x/n]^n$ for only $n \to \infty$ and $q \to 1$ is e_q^x becomes e^x.

Inverse function and new logarithm rule:
e_q^x inverse is $\ln_q(x) =?$

$$\Rightarrow \frac{(x^{(1-q)} - 1)}{(1-q)} =?$$

Remember:

$$y = (1 + (1-q)x)^{\frac{1}{(1-q)}}$$

Solve y; take exponent at everywhere to get inverse, $(1-q)$ exponent

$$y^{(1-q)} = [1 + (\ln_q y)]^{\frac{1}{(1-q)} \cdot (1-q)}$$

$$\Rightarrow a^{b^c} = a^{(b\times c)} \Rightarrow \left(\frac{1}{(1-q)}\right) \times (1-q) = 1$$

Inversing $\Rightarrow$

$$x^{(1-q)} = 1 + (1-q)y$$

$$\Rightarrow x^{(1-q)} - 1 = (1-q)y$$

$$\Rightarrow \frac{x^{(1-q)} - 1}{1-q} = y$$

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Definitions

q-logarithm $\Rightarrow$

$$\ln_q(x) = \frac{x^{(1-q)} - 1}{1-q}$$

q-exponential $\Rightarrow$

$$e_q^x = [1 + (1-q)x]^{\frac{1}{(1-q)}}$$

Now we will use these definitions, e_q^x and inverse $\ln_q(x)$
Let;

$$u = \ln_q(x_A) \Rightarrow x_A = e_q^u$$
$$v = \ln_q(x_B) \Rightarrow x_B = e_q^v$$
$$\Rightarrow x_A \times x_B = e_q^u \times e_q^v$$

$$x_A x_B = [1 + (1-q)u]^{\frac{1}{(1-q)}} \times [1 + (1-q)v]^{\frac{1}{(1-q)}}$$

Unify exponential form $\Rightarrow$

$$([1 + (1-q)u] \times [1 + (1-q)v])^{\frac{1}{(1-q)}}$$

Make the multiply in parenthesis $\Rightarrow$

$$(1-q)u \times [1 + (1-q)v] = (1-q)u + (1-q)^2 uv$$

Take $(1-q)$ parenthesis $\Rightarrow$

$$(1 + (1-q) \times [u + v + (1-q)uv])^{\frac{1}{(1-q)}} = e_q^z$$

If we say $z = u + v + (1-q)uv \Rightarrow$

$$x_A \cdot x_B = [1 + (1-q)z]^{\frac{1}{(1-q)}} = e_q^z$$

Put variables into equation $\Rightarrow$

$$\ln_q(x_A x_B) = \ln_q(x_A) + \ln_q(x_B) + (1-q)\ln_q(x_A)\ln_q(x_B)$$

References

[1] C. Tsallis, "Possible generalization of Boltzmann-Gibbs statistics," *Journal of Statistical Physics*, vol. 52, no. 1, pp. 479–487, 1988.

[2] C. Tsallis, *Introduction to nonextensive statistical mechanics: approaching a complex world*. Springer Science & Business Media, 2009.