Finding the Extremum of Tsallis Entropy

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1

What if we want to extremize S_q in condition $\sum_{i=1}^{W} p_i = 1$?

Let's introduce Lagrange parameter here. By Lagrange parameter, we can obtain S_q is extremized for all values of q, in the case of equiprobability $p_i = 1/W$.

maximize this function $\Rightarrow S_q = \frac{W^{1-q}-1}{1-q}$

Constraint: $\sum p_i - 1 = 0 \Rightarrow g(p_i)$

Set the Lagrange function $\Rightarrow L = S_q - \lambda g(p_i) \Rightarrow k \times \frac{(1-\sum p_i^q)}{(q-1)} - \lambda g(p_i)$

to find the maximum, we will take partial derivative for each part of L regard to p_i.

$$\begin{aligned} \partial L/\partial p_i &= 0 \\ \partial/\partial p_i \left[\frac{k}{q-1}(1-\sum p_i^q)\right] - \partial/\partial p_i \left[\lambda(\sum p_i - 1)\right] &= 0 \\ \Rightarrow \frac{k}{q-1}(-q \times p_i^{(q-1)}) - \lambda \times 1 &= 0 \\ \Rightarrow -k \times \frac{q}{q-1} \times p_i^{(q-1)} &= \lambda \end{aligned}$$

$\Rightarrow$ this tells us, $p_i^{(q-1)}$ statement, independent from i's equals to some constant so p_i's should be same.

2

$$p_1 = p_2 = \cdots = p_W = p$$
$$\sum p_i = W \times p$$

put into equation $\Rightarrow S_q = k \times (1 - \sum_{i=1}^{W} p_i^q)/(q-1)$

$$S_q = k \times \left(1 - \sum_{i=1}^{W} (1/W)^q\right)/(q-1)$$
$$S_q = k \cdot \frac{W^{1-q}-1}{1-q}$$

to continue into form of microcanonical situation the formula of $S_q \Rightarrow \frac{S_q}{k} = \frac{W^{(1-q)}-1}{(1-q)} \Rightarrow p_i = 1/W$

S_1 formula $\Rightarrow S_1 = -k \sum p_i \ln p_i$

$$S_1 = -k \sum_{i=1}^{W} ((1/W)\ln(1/W))$$

multiply with $W \Rightarrow S_1 = -k \times \underbrace{W\,[(1/W)}_{\text{this will be 1}}\,\ln(1/W)]$

$\Rightarrow$ regard to logarithm rule it will be

$$S_1 = -k \ln W$$

we defined W as in form S_1.

3

Let's put it into another form:

$$\frac{S_1}{k} = \ln W \Rightarrow e^{(S_1/k)} = W$$

remember

$$\frac{S_q}{k} = \frac{W^{(1-q)} - 1}{1-q} \Rightarrow \frac{e^{((S_1/k)\times(1-q))} - 1}{1-q}$$

$$\frac{S_q}{k} = \frac{e^{((1-q)S_1/k)} - 1}{1-q}$$

$\Rightarrow$ This tells us, S_q entropy not only defined by p_i probabilities but at the same time, it can be defined by system's standard entropy S_1 $(q = 1)$.

References

[1] C. E. Shannon, "A Mathematical Theory of Communication," *Bell System Technical Journal*, vol. 27, no. 3, pp. 379–423, 1948.

[2] C. Tsallis, "Possible generalization of Boltzmann-Gibbs statistics," *Journal of Statistical Physics*, vol. 52, no. 1, pp. 479–487, 1988.